On Nonlinear Feedback Strategies in Differential Games with Multiple State Variables

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Abstract

Relying on a method used in Wirl (2010) and Colombo and Labrecciosa (2013a), I illustrate a linear-quadratic linear state game with multiple states, which lends itself to be solved for nonlinear feedback strategies as if each player were facing a single state equation. Then, I also show that in general this solution does not coincide with that delivered by the method based on undetermined coefficients.

Keywords: differential games; nonlinear feedback strategies; state variable; linear state game

JEL Codes: C73, L13

1 Introduction

As is well known, the characterization of nonlinear feedback strategies in differential games has been carried out in games featuring a single state variable, since (at least) Tsutsui and Mino (1990), and a global analysis of the same problem has appeared in Rowat (2007). These two approaches have been applied in different contexts (see Shimomura, 1991; Dockner and Long, 1993; Dockner and Sorger, 1996; Itaya and Shimomura, 2001; and Rubio and Casino, among others). The problem jeopardising the design of nonlinear feedback equilibria with at least two states is essentially due to the apparent impossibility of dealing with cross partial derivatives of the value functions.

What I would like to illustrate (and discuss) in this paper is the possibility of attaining a fully analytical characterization of nonlinear feedback strategies in a particular class of differential games, i.e., those whose form is simultaneously state-linear and linear quadratic. As we shall see in the remainder, this class of differential games is indeed a relevant one and has been intensively adopted in applications to environmental and resource economics.

Additionally, the sparkle of the idea that a game of this type may lend itself to characterise nonlinear feedback equilibria, the presence of many states notwithstanding, comes a relatively recent paper in resource economics (Colombo and Labrecciosa, 2013a) featuring exactly the aforementioned structure. In what follows, I will show that the amount of symmetry used in Colombo and Labrecciosa (2013a) and appearing earlier in Wirl (2010) does the job of simplifying the resulting HJB equation to such an extent that all of the cross partial derivatives of the players' Bellman equations collapse to zero, thereby allowing one to solve the game *as if* each player faced a single state dynamics, and therefore delivering the continuum of nonlinear strategies. Initially, the game has an anonymous nature, but two explicit examples are briefly mentioned after the exposition of the main results.

This is complemented by a cautionary note illustrating that, in general, we may not expect the above approach to deliver the same solutions as the traditional method based upon undetermined parameters. Adopting the latter, indeed, one finds that the cross partial derivatives of value functions do not collapse.

The remainder of the paper is organised as follows. The structure of the game is laid out in section 2. The symmetric solution delivering both linear and nonlinear feedback equilibria, together with examples, is in section 3. The discussion about the lack of coincidence between this approach and the usual one based on undetermined coefficients is in section 4. Section 5 offers a few final remarks.

2 Problem statement

Consider first the structure of the differential game which has been used thus far to characterise the continuum of nonlinear feedback strategies. Let $X(t)$ be the single state variable common to the set of players $\mathcal{N} = 1, 2, 3, ...n$. Each of them is assigned a single control $u_i(t)$ to be chosen in the compact set $\mathcal{U}$ common to all players. The state equation is

$$\dot{X} = \delta X(t) - \alpha U(t) \tag{1}$$

where $U = \sum_{i=1}^{n} u_i(t)$ and (α, δ) are constant parameters, whose signs I will leave unspecified for the moment. Let $\pi_i(t) = \left[A - u_i(t) - \sum_{j \neq i} u_j(t)\right] u_i(t)$ be the instantaneous payoff function of player i. All players use the same constant discount factor $\rho > 0$, and the HJB equation of player i is

$$\rho V_i(X(t)) = \max_{u_i(t)} \left\{ \pi_i(t) + \frac{\partial V_i(X(t))}{\partial X(t)} \cdot \dot{X} \right\} \tag{2}$$

so that the resulting first order condition (FOC)

$$\frac{\partial \pi_i(t)}{\partial u_i(t)} + \frac{\partial V_i(X(t))}{\partial X(t)} \cdot \frac{\partial \dot{X}}{\partial u_i(t)} = 0 \tag{3}$$

can be solved w.r.t. either $u_i(t)$, if this is admissible (for instance, if it is positive - which is the obvious requirement in games in which $u_i(t)$ is a quantity) or $\partial V_i(X(t))/\partial X(t)$, otherwise. Then, one may follow the alternative routes illustrated in Tsutsui and Mino (1990) or Rowat (2007) and characterise linear and nonlinear feedback strategies.

What if there exist n states $x_i(t)$? We are accustomed to think that the presence of two states suffices to prevent us from solving the nonlinear feedback game, for the following reason. Suppose the generic state dynamics is

$$\dot{x}_i = \delta x_i(t) - \alpha U(t) \tag{4}$$

so that the HJB equation of player i becomes

$$\rho V_i(\mathbf{x}(t)) = \max_{u_i(t)} \left\{ \pi_i(t) + \frac{\partial V_i(\mathbf{x}(t))}{\partial x_i(t)} \cdot \dot{x}_i + \sum_{j \neq i} \frac{\partial V_i(\mathbf{x}(t))}{\partial x_j(t)} \cdot \dot{x}_j \right\} \tag{5}$$

where $\mathbf{x}(t) \equiv (x_1(t), x_2(t), ...x_n(t))$ is the vector of state variables. The resulting FOC is

$$\frac{\partial \pi_i(t)}{\partial u_i(t)} + \frac{\partial V_i(\mathbf{x}(t))}{\partial x_i(t)} \cdot \frac{\partial \dot{x}_i}{\partial u_i(t)} + \sum_{j \neq i} \frac{\partial V_i(\mathbf{x}(t))}{\partial x_j(t)} \cdot \frac{\partial \dot{x}_j}{\partial u_i(t)} = 0 \tag{6}$$

and this implies that we are missing a second condition to deal with the $n-1$ (eventually symmetric) partial derivatives $\partial V_i(\mathbf{x}(t))/\partial x_j(t)$. In the next section, we are about to examine a special case of the present setup in which the nonlinear feedback solution can indeed be carried out, the presence of n states notwithstanding.

3 A special case

Admittedly, what follows is a specific formulation of the game, but still a relevant one, as the examples are going to illustrate. In the remainder, the time argument is omitted for the sake of brevity. If $\partial \dot{x}_i / \partial u_j = 0$ for all $j \neq i$, the generic state dynamics is

$$\dot{x}_i = \delta x_i - \alpha u_i \tag{7}$$

Therefore, although the Bellman equation is obviously defined as in (5), the FOC pertaining to player i reads as follows:

$$\frac{\partial \pi_i}{\partial u_i} + \frac{\partial V_i (\mathbf{x})}{\partial x_i} \cdot \frac{\partial \dot{x}_i}{\partial u_i} = 0 \tag{8}$$

which does not feature any of the $n-1$ partial derivatives $\partial V_i (\mathbf{x}) / \partial x_j$. At this point, one can stress that (8) has the same structure (if not the same nature) as (3), which pertains to the single-state game.

Moreover, we may formulate a conjecture according to which, at all times, $\partial V_i (\mathbf{x}) / \partial x_j = 0$ for all $j \neq i$, on the basis of the fact that the present model defines a linear state game, with a linear-quadratic form and independent state dynamics, surely delivering a degenerate feedback Nash equilibrium under open-loop information.[1] This requires solving the following problem under simultaneous play at all times:

$$\max_{u_i} \mathcal{H}_i = e^{-\rho t} \left(\pi_i + \lambda_{ii} \dot{x}_i + \sum_{j \neq i} \lambda_{ij} \dot{x}_j \right) \tag{9}$$

where $\lambda_{ij} = e^{\rho t} \mu_{ij}$, μ_{ij} being the costate variable attached by player i to state

[1] On differential games delivering degenerate feedback solutions under open-loop rules, see Fershtman (1987), Mehlmann (1988), Dockner *et al.* (2000), Celini et al. (2005) and Lambertini (2018), *inter alia*.

j. The FOC on control u_i and the associated costate equations are

$$\frac{\partial \mathcal{H}_i}{\partial u_i} = e^{-\rho t}\left(A - 2u_i - \sum_{j \neq i} u_j + \alpha\lambda_{ii}\right) = 0 \tag{10}$$

$$\dot{\lambda}_{ii} = \lambda_{ii}(\rho - \delta) \tag{11}$$

$$\dot{\lambda}_{ij} = \lambda_{ij}(\rho - \delta) \tag{12}$$

Observe that (11-12) are differential equations in separable admitting the nil solution at every instant, in particular $\lambda_{ij} = 0$. It is also worth noting that, to guarantee the strong time consistency of the open-loop strategy, it is not necessary to set $\lambda_{ii} = 0$ because $\partial\lambda_{ii}/\partial x_i = 0$ since (11) is independent of the state due to the linear state nature of the game, and therefore also

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_i}{\partial \lambda_{ii}} \cdot \frac{\partial \lambda_{ii}}{\partial x_i} = 0 \tag{13}$$

Having quickly looked at the open-loop game, we may reformulate the above conjecture as follows:

Conjecture 1 *If the game is a linear state and linear-quadratic one, and the n states follow separated dynamics, any costates λ_{ij} are nil at all times for all $j \neq i$ while λ_{ii} may or may not be nil. This suggests that (i) the relevant value function and HJB equation of player i can be rewritten as if this player faced a single state dynamics describing the evolution of x_i :*

$$\rho V_i(x_i) = \max_{u_i}\left(\pi_i + \frac{\partial V_i(x_i)}{\partial x_i} \cdot \dot{x}_i\right)$$

and consequently (ii) the continuum of nonlinear feedback strategies can be characterised on the basis of the above HJB, as if the game were featuring a single state variable.

To the best of my knowledge, a single example showing the property stated in the first part of the above conjecture appears in a game featuring the extraction of a renewable resource from a set of parcelised pools solved for linear feedback strategies only by Colombo and Labrecciosa (2013), while part (ii) has never been discussed in the extant literature on the matter.

3.1 Solving the game

The first step consists in defining the Bellman equation of player i as

$$\rho V_i(\mathbf{x}) = \max_{u_i} \left\{ \left(A - u_i - \sum_{j \neq i} u_j \right) u_i + \right. \tag{14}$$

$$\left. \frac{\partial V_i(\mathbf{x})}{\partial x_i} (\delta x_i - \alpha u_i) + \sum_{j \neq i} \frac{\partial V_i(\mathbf{x})}{\partial x_j} (\delta x_j - \alpha u_j) \right\}$$

and prove that $\partial V_i(\mathbf{x}) / \partial x_j = 0$ for all $j \neq i$. To this purpose, we may focus on the method of undetermined coefficients to characterise linear feedback strategies. Then, by writing (8) explicitly,

$$A - 2u_i - \sum_{j \neq i} u_j - \alpha \frac{\partial V_i(\mathbf{x})}{\partial x_i} = 0 \tag{15}$$

we see that the instantaneous best reply function of player i is

$$u_i^*(U_{-i}) = \max \left\{ \frac{A - \sum_{j \neq i} u_j - \alpha \cdot \partial V_i(\mathbf{x}) / \partial x_i}{2}, 0 \right\}, \tag{16}$$

analogous to expression (7) in Colombo and Labrecciosa (2013a, p. 842). Additionally, we may stipulate that the value function takes the following linear-quadratic form:

$$V_i(\mathbf{x}) = \frac{1}{2} \left(\varepsilon_1 x_i^2 + \varepsilon_2 x^2 + \varepsilon_3 x_i x \right) + \varepsilon_4 x_i + \varepsilon_5 x + \varepsilon_6 \tag{17}$$

as in Colombo and Labrecciosa (2013a), who use the same specification of the value function as in Wirl (2010): in (17), x stands for any generic rival's state variable x_j's, for all $j \neq i$. Consequently,

$$\frac{\partial V_i(\mathbf{x})}{\partial x_i} = \varepsilon_1 x_i + \varepsilon_3 x + \varepsilon_4 \tag{18}$$

Imposing symmetry, (15) delivers the optimal control of any individual firm:

$$u^* = \max\left\{\frac{A - \alpha \cdot \partial V(\mathbf{x})/\partial x}{n+1}, 0\right\} \tag{19}$$

In the case in which $A/\alpha > \partial V(\mathbf{x})/\partial x$, we may plug the optimal control $u^* = [A - \alpha \cdot \partial V(\mathbf{x})/\partial x]/(n+1)$ back into the HJB equation (14) and solve the resulting system of Riccati equations to obtain:

$$\varepsilon_6 = \frac{\alpha^2 \varepsilon_4^2 - A(A - 2\alpha\varepsilon_4)}{\rho(n+1)^2} \tag{20}$$

$$\varepsilon_2 = \varepsilon_3 = \varepsilon_5 = 0 \tag{21}$$

$$\varepsilon_4 = \frac{2A\alpha\varepsilon_1}{2\alpha^2\varepsilon_1 + (n+1)^2(\delta - \rho)} \tag{22}$$

$$\varepsilon_{11} = 0\,;\ \varepsilon_{12} = -\frac{(n+1)^2(2\delta - \rho)}{2\alpha^2} \tag{23}$$

Clearly, if $\varepsilon_1 = \varepsilon_{11}$, the resulting optimal control is $u_{OL} = A/(n+1)$, i.e., the open-loop (or, degenerate feedback) one. If instead $\varepsilon_1 = \varepsilon_{12}$, the resulting equilibrium strategy is a proper linear feedback one:

$$u_{LF} = \frac{2A\alpha(\rho - \delta) + \delta(2\delta - \rho)(n+1)^2 x}{2\alpha\delta(n+1)} \tag{24}$$

which indeed coincides with the expression of the optimal control in Colombo and Labreccciosa (2013a, Prop. 1, p. 843) if $A = \alpha = 1$.

Solutions (21) imply that the $n-1$ states pertaining to rivals have no impact on the feedback problem solved by any of the players, as $\partial V_i(\mathbf{x})/\partial x_j = 0$

for all $j \neq i$. Indeed, it is easily shown that the foregoing analysis can be replicated one-to-one by redefining the HJB equation of player i simply as

$$\rho V_i(x_i) = \max_{u_i} \left\{ \left(A - u_i - \sum_{j \neq i} u_j \right) u_i + \frac{\partial V_i(x_i)}{\partial x_i} (\delta x_i - \alpha u_i) \right\} \tag{25}$$

and the associated value function as

$$V_i(x_i) = \frac{1}{2}\eta_1 x_i^2 + \eta_2 x_i + \eta_3 \tag{26}$$

with the related system of Riccati equations delivering

$$\eta_3 = \varepsilon_6 \, ; \, \eta_2 = \varepsilon_4 \, ; \, \eta_{11} = \varepsilon_{11} \, ; \, \eta_{12} = \varepsilon_{12} \tag{27}$$

and consequently the same expressions for u_{OL} and u_{LF} as above.

So, if indeed individual player's standpoint in a game like this can be specified through (25-26), this creates the possibility of characterising the continuum of nonlinear feedback strategies which, if $\partial V_i(\mathbf{x}) / \partial x_j \neq 0$, would remain out of reach as we have been accustomed to think thus far.

The remainder of this section is devoted to the derivation of the nonlinear feedback solutions following the same procedure as in Fujiwara (2008, 2009), i.e., leaving aside the full details of the global analysis based on Rowat (2007), as this is not the focus of the present paper.[2] The FOC obtaining from (25), after imposing symmetry across players, can be solve w.r.t. the partial derivative of the value function,

$$\frac{\partial V(x)}{\partial x} = \frac{A - (n+1)u}{\alpha} \tag{28}$$

[2]Of course, this could be easily added but its attainment, in some sense, is redundant as the point here is whether nonlinear strategies can be described or not in a game *a priori* defined in terms of multiple states. The procedure can be found in Rowat (2007) and is replicated in Lambertini (2016, 2018), Lambertini and Mantovani (2016) and Lambertini and Palestini (2018).

Substituting (28) into (25) and solving it w.r.t. $V(x)$, we obtain

$$V(x) = \frac{\alpha u^2(x) + \delta[A - (n+1)u(x)]x}{\alpha\rho} \tag{29}$$

where it clearly appears that we expect the generic control to depend on x (with the obvious exception of the open-loop solution). Differentiating both sides w.r.t. x, we have

$$\frac{\partial V(x)}{\partial x} = \frac{\delta[A - x(n+1)u'(x)] - u(x)[\delta(n+1) - 2\alpha u'(x)]}{\alpha\rho} \tag{30}$$

and the r.h.s. of (30) must be equal to the r.h.s. of (28), whereby we identify the partial derivative of the individual control w.r.t. the state:

$$u'(x) = \frac{[A - (n+1)u(x)](\delta - \rho)}{\delta(n+1)x - 2\alpha u(x)} \tag{31}$$

The above equation implicitly defines any nonlinear feedback strategy, including as special cases the linear ones, which obtains by specifying $u(x) = a + bx$, so that $u'(x) = b$ and (31) engenders the following system:

$$\begin{gathered} A(\delta - \rho) + a[2b\alpha + (\rho - \delta)(n+1)] = 0 \\ b[2b\alpha + (\rho - \delta)(n+1)] = 0 \end{gathered} \tag{32}$$

which is solved by

$$\begin{gathered} \widehat{a} = \frac{A}{n+1}; \widehat{b} = 0 \\ \widetilde{a} = \frac{A(\rho - \delta)}{\delta(n+1)}; \widetilde{b} = \frac{(n+1)(2\delta - \rho)}{2\alpha} \end{gathered} \tag{33}$$

The pair $\left(\widehat{a}, \widehat{b}\right)$ identifies u_{OL}, while the pair $\left(\widetilde{a}, \widetilde{b}\right)$ generates u_{LF}. Again from (31), it is also worth noting that looking for u_{OL} is analogous to imposing $u'(x) = 0$.

Finally, we may single out the tangency point which partitions the set of steady states into two subsets containing stable and unstable ones. The steady state locus being $u^{ss} = \delta x/\alpha$, we have $u'(x) = \delta/\alpha$ and therefore (31) must be rewritten as follows:

$$\frac{\delta}{\alpha} = \frac{[A - (n+1)\,\delta x/\alpha]\,(\delta - \rho)}{\delta\,(n+1)\,x - 2\alpha\delta x/\alpha} \tag{34}$$

and this equation is solved by

$$x_T = \frac{A\alpha\,(\rho - \delta)}{\alpha^2\rho\,(n+1) - \delta\,[\alpha^2\,(n-1) + \delta\,(n+1)]} \tag{35}$$

where subscript T mnemonics for the tangency condition. The associated level of the control is $u_T = \delta x_T/\alpha$.

The foregoing analysis is summarised in the following

Proposition 2 *Consider a differential game in linear-quadratic form, with n players, a vector of n controls u and a vector of n states x, with any single player i associated with a single pair $(x_i, u_i) \neq (x_j, u_j)$. If $\partial V_i(\mathbf{x})/\partial x_j = 0$ for all $j \neq i$, then it is possible to characterise the continuum of nonlinear feedback strategies together with the linear ones.*

There remains to say a few words about the stability properties of these feedback equilibria, which basically depends upon the signs of α and δ. This aspect is quickly dealt with in the next subsection.

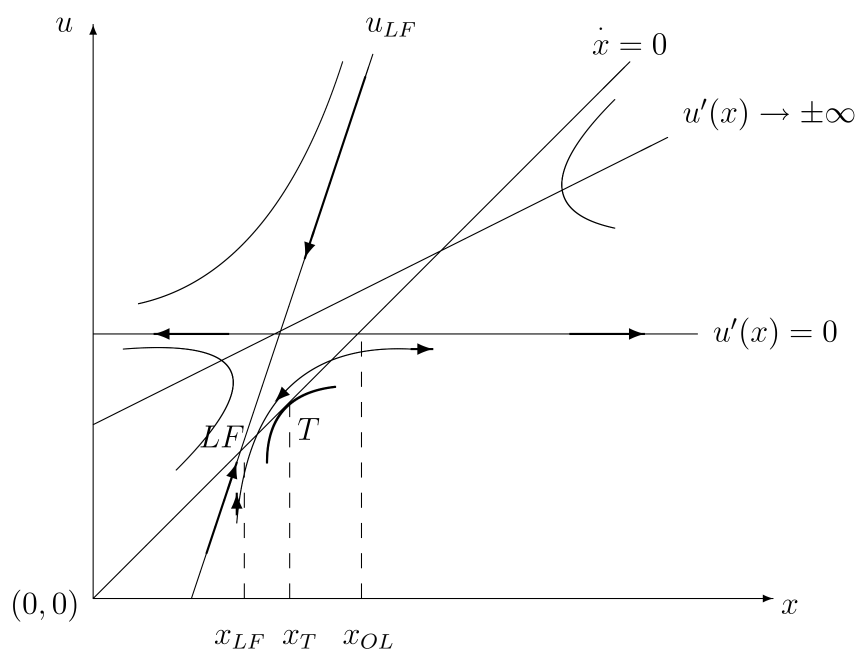
3.2 Examples

Specifying the signs of α and δ amounts to put an explicit label onto the state dynamics and therefore also the nature of the state itself. Suppose $\alpha, \delta > 0$: if so, the state equation (1) is a linear approximation of the Lotka (1925) - Volterra (1931) model describing the logistic growth of a natural species. This simplified dynamics appears in Fujiwara (2008), Colombo and Labrecciosa (2013a); Lambertini and Leitmann (2013); Lambertini and Mantovani

(2014, 2016); and Lambertini (2016), among others.[3] In combination with the objective function π_i, this implies that we are talking about a Cournot industry extracting a renewable resource.

In this case, the phase diagram looks as in Figure 1, where the arrows appearing along the nonlinear strategies illustrate the stability of the steady state points belonging to the segment LFT. the diagram also portrays the non-invertibility line $u'(x) \rightarrow \pm\infty$.

Figure 1 The phase diagram, $\alpha, \delta > 0$



In the opposite scenario, in which $\alpha, \delta < 0$, the model may describe a

[3] A a piecewise linear approximation of the original state equation appears instead in Benchekroun (2003, 2008) and Colombo and Labrecciosa (2013b, 2015). The characterization of nonlinear strategies outlined above can be applied to this alternative setup as well.

Cournot industry with unregulated polluting emissions generated by either production or consumption, or both, as in Benchekroun and Long (1998) and Fujiwara (2009), *inter alia*. Flipping the sign of both parameters implies that the direction of all the arrows flips over as well, and all the stable steady state equilibria belong to the segment of the steady state locus between T and the intersection with the flat line identifying the open-loop (or degenerate feedback) strategy. In both cases, the tangency point is semi-stable.

4 A plausible objection

The foregoing perspective, intuitively, looks quite appealing, although being drawn only for a linear-state linear-quadratic game. Yet, one may formulate a sound - but not necessarily compelling - argument against its validity and therefore its adoption. The objection sounds as follows: if the above procedure is correct and reliable, an analogous conclusion should arise from the traditional solution method based upon undetermined coefficients, *without imposing any symmetry condition.*

To clarify the matter, we may restrict our attention to the case of two players, having thus the HJB equation

$$\rho V_i(x_i, x_j) = \max_{u_i} \left\{ (A - u_i - u_j) u_i + \frac{\partial V_i(x_i, x_j)}{\partial x_i} (\delta x_i - \alpha u_i) + \frac{\partial V_i(x_i, x_j)}{\partial x_j} (\delta x_j - \alpha u_j) \right\} \quad (36)$$

with $i, j = 1, 2$ and $j \neq i$, and the following value function

$$V_i(x_i, x_j) = \varepsilon_1 x_i^2 + \varepsilon_2 x_j^2 + \varepsilon_3 x_i x_j + \varepsilon_4 x_i + \varepsilon_5 x_j + \varepsilon_6 \quad (37)$$

As a result, the optimal control, whenever positive, is

$$u_i^* = \frac{A - \alpha \left[2\partial V_i(\cdot) / \partial x_i - \partial V_i(\cdot) / \partial x_j \right]}{3} \quad (38)$$

Simplifying (36), one obtains the usual system of six Riccati equations, the first of which being independent of states, satisfied by

$$\varepsilon_6 = \frac{(A - \alpha\varepsilon_4)\left[A - \alpha\left(\varepsilon_4 + 3\varepsilon_5\right)\right]}{9\rho} \tag{39}$$

Imposing $\varepsilon_2 = \varepsilon_3 = \varepsilon_5 = 0$ and solving the Riccati equation pertaining to ε_4, we get

$$\varepsilon_4 = \frac{8A\alpha\varepsilon_1}{8\alpha^2\varepsilon_1 + 9\left(\delta - \rho\right)} \tag{40}$$

and then we remain with the last Riccati equation:

$$\varepsilon_1\left[9\rho - 2\left(8\alpha^2\varepsilon_1 + 9\delta\right)\right] = 0 \tag{41}$$

whose solutions are

$$\varepsilon_{11} = 0\,;\; \varepsilon_{12} = -\frac{9\left(2\delta - \rho\right)}{16\alpha^2} \tag{42}$$

Posing $\varepsilon_1 = \varepsilon_{11} = 0$, we obtain the open-loop solution and, indeed, (36) is solved, alongside with the system of Riccati equations. This, however, does not happen if we pose $\varepsilon_1 = \varepsilon_{12}$, whereby the three equations pertaining to x_j, $x_i x_j$ and x_j^2 are not satisfied, since the related polynomials simply as follows:

$$\begin{array}{c} \dfrac{9A\left(2\delta - \rho\right)\left(\rho - \delta\right)}{2\alpha\rho} \\ \dfrac{81\left(2\delta - \rho\right)^2}{16\alpha^2} \\ -\dfrac{81\left(2\delta - \rho\right)^2}{64\alpha^2} \end{array} \tag{43}$$

In a nutshell, both methods work equally well as far as the degenerate linear feedback solution is concerned, while the imposition of symmetry is not delivering the same solution one would attain by taking the traditional route.

In a nutshell, the issue boils down to this: the use of a certain degree of symmetry (i) solves the HJB equation and (ii) permits the identification of the nonlinear feedback strategies as in games with a single state variable, but (iii) this method coincides with the usual one - which makes use of no symmetry at all - only for the open-loop equilibrium strategy. Accordingly, (iv) it may be used for the design of nonlinear feedback strategies in games in which the stable linear feedback solution is the open-loop one (which, in terms of the above examples, means that we may take that route in oligopoly games with polluting emissions). There remains, however, that both methods do solve the HJB equation of the individual firm, and therefore both are, in line of principle, delivering a solution of the initial problem defining the differential game. Can or must we really leave aside the method based on symmetry because its outcome does not coincide, in general, with that of the traditional procedure based on undetermined parameters?

5 Concluding remarks

The matter discussed in this paper has an ambiguous taste. On the one hand, a certain amount of symmetry permits to treat a problem that we have considered intractable for a long time. On the other, recapitulating what we know from the tradition of differential game theory, we may feel obliged to step back a bit, because the standard solution relying on undetermined coefficients and no symmetry yields a different solution. Yet, both methods produce a solution to the problem defined at the outset. My personal view is that the real question is the following: given that we are talking about firms, or policy makers or some other set of agents, what really matters is what frame of mind we shall attribute to a class of players - as, in fact, the two methods (more or less implicitly) do refer to two alternative standpoints which may be taken by any individual player when looking at her/his opponents in a given environment.

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